Loschmidt echo and Berry phase around degeneracies in nonlinear systems

X. X. Yi, X. L. Huang, and W. Wang

School of Physics and Optoelectronic Technology, Dalian University of Technology, Dalian 116024, China

(Received 5 January 2008; revised manuscript received 14 April 2008; published 22 May 2008)

We study the Loschmidt echo and Berry phase in a nonlinear system transported around a double and a triple degeneracy. The nonlinearity of the system makes this Berry phase different from that in linear systems. We propose a witness of nonlinearity for the nonlinear system and show its dependence on the parameters of the system, taking the standard Landau-Zener model as an example. We calculate the Loschmidt echo (LE) or quantum fidelity in the quantum dynamics under perturbations around the degeneracy, and establish a connection between the LE and the witness of nonlinearity. These phases can be observed with current experimental technology in a nonlinear resonator.

I. INTRODUCTION

Since Berry’s introduction [1] of the adiabatic geometric phase, a large number of articles have appeared about the theoretical foundations, physical applications, and experimental manifestations of geometric phases [2,3]. Although there are by now hundreds of papers on geometric phases, there were no studies on this subject in nonlinear systems until the seminal works [4,5], due to the lack of orthogonality of their Hamiltonian eigenstates and the linear superposition principle.

Singularities in mathematics are often associated with specific effects in physics. For instance, topological singularities such as diabolic points are associated with a specific phase behavior of the wave function [6,7]. Exceptional points, another type of topological singularity, are associated with level repulsions and can affect Rabi oscillations [8]. For a linear system, it has been shown that the eigenfunction of a real Hamiltonian can acquire a $\pi$ geometric phase (a sign change of the wave function) when it is transformed around a certain type of degeneracy. This sign change was found in the late 19th century, but its significance to physics was not realized until Longuet-Higgins and co-workers showed its existence in molecular physics [9]. The latter insight led to the notion of the molecular Aharonov-Bohm effect [10] which has attracted a lot of attention both experimentally and theoretically in recent years [11–13]. For nonlinear systems, we may have additional eigenvalues and eigenvectors beyond the dimension of its Hilbert space. The degeneracy in this system is also different from that in linear systems; for example, the eigenfunctions at the degeneracy point may not be orthogonal to each other in general. This makes the Berry phase around a degeneracy in a nonlinear system different from that in linear systems.

In this paper, we shall study the Berry phase in the vicinity of double degeneracies in nonlinear systems. A general expression for the Berry phase will be given and discussed. The results show that the Berry phase around the degeneracy significantly depends on the nonlinearity characterized by the overlap of the degenerate eigenvectors at the degeneracy point. This overlap will be defined as a witness of nonlinearity, which is found to be related to the Loschmidt echo in the quantum dynamics around the degeneracy. As an example, we calculate the Berry phase and the witness of nonlinearity around the double degeneracy in the standard Landau-Zener model. An extension from double to triple degeneracy is also presented and discussed.

The paper is organized as follows. In Sec. II, we examine the Berry phase around a double degeneracy in a nonlinear system, and propose a witness for the nonlinearity. A connection between the witness and the Berry phase is established, and the Loschmidt echo is calculated and discussed. In Sec. III, we present an example to illustrate the dependence of the Berry phase on the system parameters in the Landau-Zener model. An extension of this study from double to triple degeneracy is presented in Sec. IV. Finally, a discussion of the experimental observation of our prediction and the conclusion are presented in Sec. V.

II. THE BERRY PHASE AROUND A DOUBLE DEGENERACY AND A WITNESS OF NONLINEARITY

We start by recalling the calculation of the Berry phase in a general system (linear or nonlinear) with Hamiltonian $H(X)$ [5], where $X=(X_1, X_2, \ldots, X_m)$ is a vector parameter that the system depends on. For a quantum system starting at an eigenstate $|\Psi_n(X)\rangle$ defined by

$$H(X)|\Psi_n(X)\rangle = E_n(X)|\Psi_n(X)\rangle,$$

the state at time $t$ may be written as $|\Psi(t)|=e^{-i\phi}|\Psi_n(X)\rangle$ when $X$ changes adiabatically. The Berry phase in this case is

$$\gamma_n = i\oint_c \frac{\langle \Psi_n(X) | \partial_i \partial_j | \Psi_n(X) \rangle}{\langle \Psi_n(X) | \Psi_n(X) \rangle} dX.$$

Equation (2) is valid for both linear and nonlinear quantum systems as long as the system is initially in an eigenstate of the system Hamiltonian. Equation (2) simplifies after applying the standard normalization $\langle \Psi_n(X) | \Psi_n(X) \rangle = 1$. Consider a double degeneracy [14] $E_{n1} = E_{n2}$, we denote by $X_d$ the degenerate point in the parameter space, i.e., $E_n(X_d) = E_{n1} = E_{n2}$. The eigenvectors at this point $|\Psi_n^{d1}\rangle$ and $|\Psi_n^{d2}\rangle$ are chosen to be normalized $\langle \Psi_n^{d1} | \Psi_n^{d2} \rangle = 1$. The orthogonality condition $\langle \Psi_n^{d1} | \Psi_n^{d1} \rangle = 0$ holds for linear systems, but it is not sat-
isfied for nonlinear systems in general. As we shall show, this makes the Berry phase around the double degeneracy different for linear and nonlinear systems.

Consider a small circular loop around the double degeneracy point \(X_d\), \(C = \{X(t) = X_d + \delta X(t)\}\) with \(\delta X(0) = \delta X(T)\). Here \(t \in [0, T]\) represents the time and \(T\) stands for the duration of the cyclic evolution. Suppose that the perturbation theory holds around the double degeneracy point \(X_d\), the eigenvectors and eigenvalues at the point \(X(t)\) take the asymptotic forms

\[
|\Psi_n(X(t))\rangle = \cos \frac{\theta}{2} |\Psi_n^d\rangle + \sin \frac{\theta}{2} e^{i\phi} |\Psi_{n+1}^d\rangle,
\]

\[
|\Psi_{n+1}(X(t))\rangle = \sin \frac{\theta}{2} |\Psi_n^d\rangle - \cos \frac{\theta}{2} e^{-i\phi} |\Psi_{n+1}^d\rangle,
\]

\[
E_j(X(t)) = E_d + \delta E_j, \quad j = n, n+1,
\]

where \(\delta E_j\) are eigenvalues of the following matrix:

\[
\delta H = \begin{pmatrix} \delta H_{n,n} & \delta H_{n,n+1} \\ \delta H_{n,n+1} & \delta H_{n+1,n+1} \end{pmatrix}.
\]

Here \(\delta H_{n,n} = \langle \Psi_n^d \left| H_{\alpha,n} \right| \Psi_n^d \rangle\), \(\delta H_{n,n+1} = \langle \Psi_n^d \left| H_{\alpha,n} \right| \Psi_{n+1}^d \rangle\), \(\alpha, \beta = n, n+1\). After some simple manipulations, we find that

\[
\delta E_{n,n+1} = \frac{\delta H_{n,n} + \delta H_{n,n+1} + 1}{2} \pm \sqrt{\frac{1}{4} (\delta H_{n,n} - \delta H_{n,n+1})^2 + |\delta H_{n,n+1}|^2},
\]

\[
\cos \theta = \frac{\delta H_{n,n} - \delta H_{n,n+1}}{\sqrt{(\delta H_{n,n} - \delta H_{n,n+1})^2 + 4|\delta H_{n,n+1}|^2}},
\]

\[
\phi = \arg(\delta H_{n,n+1}).
\]

By using Eq. (2), we arrive at an asymptotic expression for the Berry phase,

\[
\gamma_{n,n+1} = \frac{1}{2} \int \frac{1}{\Delta_\pm} (1 \mp \cos \theta) d\phi \pm \frac{1}{2} \int \frac{|\langle \Psi_n^d | \Psi_{n+1}^d \rangle| \sin \theta}{\Delta_\pm} d\phi.
\]

Here \(\Delta_\pm = 1 \pm |\langle \Psi_n^d | \Psi_{n+1}^d \rangle| \sin \theta\), and \(\theta\) was assumed to be \(\phi\) independent. These phases simplify,

\[
\gamma_n = \frac{\pi}{\Delta_+} (1 - \cos \theta) + \frac{\pi}{\Delta_+} |\langle \Psi_n^d | \Psi_{n+1}^d \rangle| \sin \theta,
\]

\[
\gamma_{n+1} = \frac{\pi}{\Delta_-} (1 + \cos \theta) - \frac{\pi}{\Delta_-} |\langle \Psi_n^d | \Psi_{n+1}^d \rangle| \sin \theta,
\]

when \(|\langle \Psi_n^d | \Psi_{n+1}^d \rangle|\) is independent of \(\phi\). For linear systems, the eigenvectors \(|\Psi_n^d\rangle\) and \(|\Psi_{n+1}^d\rangle\) are orthogonal, leading the Berry phases to become \(\gamma_n = \frac{\Omega}{2}\) and \(\gamma_{n+1} = -\frac{\Omega}{2}\), where \(\Omega\) is the solid angle subtended by \(C\) on the unit sphere in the space. For a real Hamiltonian that describes reversible systems, \(|\Psi_n^d\rangle\) and \(|\Psi_{n+1}^d\rangle\) can be chosen to be real such that the cycle \(C\) around the degeneracy point \(X_d\) lies in a plane [15]. As a consequence, we have \(\gamma_n = \gamma_{n+1} = \pi\) if \(C\) makes a single turn around the degeneracy point, while the Berry phase is zero if the degeneracy point lies outside the cycle [1].

For nonlinear systems, however, the eigenvectors \(|\Psi_n^d\rangle\) and \(|\Psi_{n+1}^d\rangle\) are not orthogonal in general. Assuming \(|\langle \Psi_n^d | \Psi_{n+1}^d \rangle|\) to be independent of \(\phi\) and very small \((|\langle \Psi_n^d | \Psi_{n+1}^d \rangle| \ll 1)\), the Berry phases in this case reduce to

\[
\gamma_{n,n+1} = \pm \frac{\Omega}{2} + \left(\frac{\Omega_e}{4} + \frac{\pi}{2}\right)|\langle \Psi_n^d | \Psi_{n+1}^d \rangle|, \quad (8)
\]

where \(\Omega_e\) was defined as \(\Omega_e = -\frac{\phi}{2} \log(1 + \cos(\pi - 2\theta)) d\phi\), which stands for the solid angle with \(\theta = (\pi - 2\theta)\) instead of \(\theta\) in \(\Omega_e\). For \(|\langle \Psi_n^d | \Psi_{n+1}^d \rangle|\rightarrow 1\), the Berry phases reduce to \(\gamma_{n,n+1} = \frac{\Omega_e}{2} (1 \pm \cos(\pi - 2\theta)) d\phi\). This analysis shows that the Berry phase around the degeneracy point significantly depends on the overlap \(|\langle \Psi_n^d | \Psi_{n+1}^d \rangle|\), and this overlap might be related to the nonlinearity of the system. Indeed, as we shall demonstrate in the next paragraph, the overlap may be chosen as a witness of nonlinearity. This witness of nonlinearity is related to the Loschmidt echo (LE) as follows. The LE was defined as the overlap between two states that evolve from the same initial wave function \(|\Psi_0\rangle\) under two slightly different Hamiltonians \(H(X_d)\) and \(H(X_d) + \delta H\), respectively.

\[
L(t) = |\langle \Psi_0 | U(t + \delta H) U(t) | \Psi_0 \rangle|^2, \quad (9)
\]

where \(U(t) = H(X(t))\) stands for the time evolution operator corresponding to Hamiltonian \(h\). Under adiabatic evolution, the LE reads

\[
L(t) = \frac{1}{A} \left| \cos \left(\frac{\theta}{2} \sin \theta \right) \langle \Psi_n^d | \Psi_{n+1}^d \rangle \right|^2, \quad (10)
\]

where \(|\Psi_0\rangle = |\Psi_n^d\rangle\) was assumed. This result indicates that the LE, which quantifies the stability of the quantum dynamics of a system against perturbations, is closely connected to \(|\langle \Psi_n^d | \Psi_{n+1}^d \rangle|\), i.e., the overlap can be a good witness of nonlinearity. For linear systems, \(|\langle \Psi_n^d | \Psi_{n+1}^d \rangle| = 0\), leading to \(L(t) = \cos^2 \frac{\theta}{2}\). When the Hamiltonian \(H(X_d) + \delta H\) drives the system to follow the circular loop that lies in the equator of the sphere \((r, \theta, \phi)\), where \(r = \frac{1}{2} (\delta E_{n,n} - \delta E_{n,n+1})\), we have \(\cos \theta = 0\), then the LE becomes \(L(t) = \frac{1}{2}\). In this case, the Berry phases reduce to \(\gamma_{n+1} = \gamma_n = \pi\), \(\gamma_{n,n+1}\) in Eq. (6) can reproduce the Berry phase in linear systems even with nonzero overlap. This happens when \(\delta H_{n,n+1} < \delta H_{n,n} - \delta H_{n,n+1,1}\), reminiscent of the adiabatic condition in linear systems. The Loschmidt echo in this situation becomes \(L(t) = |\langle \Psi_n^d | \Psi_{n+1}^d \rangle|^2\) with \(\delta H_{n,n} < \delta H_{n,n+1}\), and \(L(t) = 1\) with \(\delta H_{n,n} > \delta H_{n,n+1}\). This feature clearly bridges the Loschmidt echo and the witness of nonlinearity and might make the witness of nonlinearity experimentally observable. The Loschmidt echo may decay in quantum systems whose classical counterparts have strong chaos with exponential instability; this decay has been well studied in [16,17]. Equation (10) sheds light on this instability from a new point of view, i.e., the overlap \(|\langle \Psi_n^d | \Psi_{n+1}^d \rangle|\). To get further insight into this Berry phase and the witness, we shall calculate the Berry
phase around the degeneracy and the witness of nonlinearity in the Landau-Zener model in the next section. This example will show clearly how the witness depends on the parameters in the model.

III. THE BERRY PHASE AROUND A DOUBLE DEGENERACY IN THE LANDAU-ZENER MODEL

As an example, we demonstrate the Berry phase $\gamma_n$ with a nonlinear two-level model as

$$\frac{\partial}{\partial t} |\psi_1\rangle = \left( \frac{R}{2} + \frac{c}{2} m + \frac{v}{2} e^{i \phi} \right) |\psi_1\rangle - \left( \frac{R}{2} - \frac{c}{2} m \right) |\psi_2\rangle,$$

(11)

where $m=|\psi_2|^2-|\psi_1|^2$, and $|\psi_1\rangle$ and $|\psi_2\rangle$ are the probability amplitudes. $v e^{i \phi}$ is the complex coupling between the two levels, $c$ stands for the nonlinear parameter that characterizes the dependence of the level energy on the populations, and $R$ is the level bias. This model can be used to describe the Josephson effect of Bose-Einstein condensates in a double-well potential [18,19]. The complex coupling can be realized in experiments with current technology [20]. To study the Berry phase, we need to find all the eigenstates $|\Psi_n\rangle$ of the nonlinear Hamiltonian

$$H = \left( \frac{R}{2} + \frac{c}{2} m + \frac{v}{2} e^{i \phi} \right) |\psi_1\rangle - \left( \frac{R}{2} - \frac{c}{2} m \right) |\psi_2\rangle.$$

(12)

By Eq. (2), we obtain the Berry phase

$$\gamma = \pi \left( 1 - \sqrt{1 - \frac{v^2}{4c^2}} \right),$$

(13)

where $E$ is one of the real roots of the equation $E^3 + cE^3 + \frac{1}{4}(c^2 - v^2 - R^2)E^2 - \frac{v^4}{16} E + \frac{v^2 c^2}{16} = 0$. To derive Eq. (13), we restricted the system to follow a path with fixed $R$, $c$, and $v$, i.e., only $\phi$ is allowed to change. It may have at most four real roots, indicating that more than two eigenstates can exist in that system [see Figs. 1(a)–1(c)]. In order to involve the Berry phase around the degeneracy point $D (R=0)$, where the first and second eigenvalues touch, we choose the lowest eigenvalue as the $E$ in Eq. (13); we plot the Berry phase as a function of $R$ and $v$ in Fig. 1(d). The nonlinear parameter $c=1$ was set for this plot. For weak nonlinearity ($u \ll c$), the Berry phase is $\pi$ at $R=0$, while it is zero for strong nonlinearity ($u \gg c$). As shown in [21] $c=v$ [Fig. 1(b)] is a critical point for the system to have more eigenvalues. For $c<v$ [Fig. 1(a)] there are two eigenvalues while there can be four eigenvalues when $c>v$ [Fig. 1(c)]. We can find this critical point in Fig. 2, where a step change along the line $R=0$ occurs in the overlap $|\langle \Psi_n | \Psi_{n+1} \rangle|$. Two observations can be made from Fig. 2. (1) As $v/c$ increases, the overlap tends to zero, indicating that the system changes from nonlinear to linear. (2) A step change in the overlap occurs at the degeneracy point $R=0$; it becomes unclear when $R$ is far from the degeneracy point. These features make the overlap a good witness for the nonlinearity and critical point.

IV. THE BERRY PHASE AROUND A TRIPLE DEGENERACY IN NONLINEAR SYSTEMS

By extending the presentation to the case of triple degeneracy, we shall show in this section that the Berry phase around a triple degeneracy will differ from that around a double degeneracy; all allowed phase changes around this triple degeneracy are given and discussed. Consider a Hamiltonian that removes the threefold degeneracy $E_{n-1} = E_n = E_{n+1}$.

$$\delta H = \begin{pmatrix}
\delta H_{n-1,n-1} & \delta H_{n-1,n} & \delta H_{n-1,n+1} \\
\delta H_{n,n-1} & 0 & 0 \\
\delta H_{n+1,n-1} & 0 & 0
\end{pmatrix},$$

(14)

where $\delta H_{a,b} \equiv \alpha, \beta = n-1, n, n+1$, is the same as in Eq. (4) but here the degeneracy is threefold, and the degenerate wave functions $|\Psi_{n-1}\rangle$, $|\Psi_n\rangle$, and $|\Psi_{n+1}\rangle$ are chosen to be real. This approximation consists of neglecting transition amplitudes between the three degenerate states and other states as well as the transitions between the $n$th and $(n+1)$th degenerate states. $\delta H_{n+1,n-1}$ and $\delta H_{n,n}$ are also assumed to be

FIG. 1. (Color online) Energy levels and the Berry phase as a function of $R$ and $v$. (a), (b), and (c) are plotted for the energy level with $v=0.2$, (b) 1, and (c) 0.5. The phase in (d) was computed in units of $\pi$. The other parameter chosen is $c=1$.

FIG. 2. (Color online) Overlap of the two degenerate wave functions. $c=1$ was set for this plot. (a)–(h) correspond to $R=0, 0.2, 0.4, 0.8, 1.4, 1.6, 1.8,$ and 2.0, respectively.
equal. This kind of Hamiltonian can be realized by treating the threefold degenerate state as a three-level Λ system, labeling one of the degenerate states as \( |e \rangle \) and the other two as \( |a \rangle \) and \( |g \rangle \). The state \( |e \rangle \) is coupled to states \( |a \rangle \) and \( |g \rangle \) with coupling constants \( \delta H_{n-1,a} \) and \( \delta H_{n-1,g} \), respectively. The Λ-type system has been intensively studied in quantum optics and recently it was used to discuss the adiabatic condition for nonlinear systems [22]. Defining \( \Omega^2 = (\delta H_{n-1,a})^2 + 4(\delta H_{n-1,a})^2 + 4(\delta H_{n-1,g})^2 \), \( \sin \theta \cos \phi = \frac{2}{\pi} \delta H_{n-1,a} \) and \( \sin \theta \sin \phi = \frac{2}{\pi} \delta H_{n-1,g} \), we can write the eigenstates and eigenvalues for \( \delta H \) as

\[
\left| \Psi_{n-1} \right\rangle = -\cos \frac{\theta}{2} (\sin \phi |\Psi_{n+1}^d \rangle + \cos \phi |\Psi_{n}^d \rangle) + \sin \frac{\theta}{2} |\Psi_{n+1} \rangle,
\]

\[
|\Psi_{n} \rangle = \cos \phi |\Psi_{n+1}^d \rangle - \sin \phi |\Psi_{n}^d \rangle,
\]

\[
|\Psi_{n+1} \rangle = \sin \frac{\theta}{2} (\sin \phi |\Psi_{n+1}^d \rangle + \cos \phi |\Psi_{n}^d \rangle) + \cos \frac{\theta}{2} |\Psi_{n+1} \rangle.
\]

(15)

Here \( \theta \) varies from 0 to \( 2\pi \); the triple degeneracy happens at \( \Omega = 0 \) [the origin of the sphere \( (r=\Omega, \theta, \phi) \)]. Equation (15) shows that \( |\Psi_{n}(\theta+\pi)\rangle = |\Psi_{n}(\theta)\rangle \), \( |\Psi_{n+1}(\theta+\pi)\rangle = |\Psi_{n+1}(\theta)\rangle \), and \( |\Psi_{n-1}(\theta+\pi)\rangle = -|\Psi_{n-1}(\theta)\rangle \), implying \( |\Psi_{n}(\theta+2\pi)\rangle = |\Psi_{n}(\theta)\rangle \), \( |\Psi_{n+1}(\theta+2\pi)\rangle = -|\Psi_{n+1}(\theta)\rangle \), and \( |\Psi_{n-1}(\theta+2\pi)\rangle = -|\Psi_{n-1}(\theta)\rangle \). In fact, the latter relation can be found straightforwardly from Eq. (15). This completes the picture when the circular path lies on the great circle of the sphere. In the case where the path lies on the equator of the sphere, namely, \( \theta = \frac{\pi}{2} \), Eq. (15) follows \( |\Psi_{n}(\phi+2\pi)\rangle = |\Psi_{n}(\phi)\rangle \), \( |\Psi_{n+1}(\phi+2\pi)\rangle = -|\Psi_{n+1}(\phi)\rangle \), and \( |\Psi_{n-1}(\phi+2\pi)\rangle = -|\Psi_{n-1}(\phi)\rangle \). So for the triple degeneracy we have two allowed adiabatic sign changes around the degeneracy, which are listed in Table I. It is worth pointing out that the Berry phases presented in this case are all for loops which enclose the point of degeneracy. By enclose we mean that the loop cannot be smoothly deformed to avoid surrounding the degeneracy point in the parameter space. This is different from that for the double degeneracy, where the Berry phase is calculated for a general circular loop around the degeneracy point.

V. DISCUSSION AND CONCLUSION

The presented prediction is within reach of recent experimental technology. The observation of the Berry phase around a triple degeneracy is available via the setup in [23] but with Kerr nonlinearity. Due to the nonlinear interaction, the triple degeneracy point may change; however, it can be relocated by measuring the spectrum of the rectangular resonator, which is a function of the shape of the resonator [represented by \((a, b)\) as shown in Fig. 3]. The geometric phase effect then can be measured as Lauber et al. did in Ref. [23] as follows, (1) After finding the degeneracy point, shift the position of the mirror at one of its corners (say \(D\)) around \((a, b)\). (2) Measure standing wave patterns via the reflected microwave intensity by the technology in [26]. The Berry phase effect can be found by comparing those patterns at the start and end points. The Berry phase around the double degeneracy can be observed in the same manner, by finding a double degeneracy instead of the triple degeneracy in the spectra of triangular resonators. Alternatively, Bose-Einstein condensates in a double-well potential may meet the requirement of observing the Berry phase around a double degeneracy, serving as the nonlinear system.

In conclusion, we have studied the Loschmidt echo and Berry phase around a degeneracy in nonlinear systems. Two degeneracies, a double degeneracy and a triple degeneracy, are considered. We have found that the Berry phase is different from that in linear systems due to nonorthogonality of the degenerate wave functions at the degeneracy point. We have also found that the overlap of the degenerate wave functions may serve as a witness of the nonlinearity of a nonlinear system. A connection between this witness and the Loschmidt echo has been established. The allowed adiabatic sign changes around the degeneracy point are presented and discussed. The above analysis is based on first-order perturbation theory, which fails in the presence of additional satellite degeneracy near the main degeneracy [24]; this problem can be solved in linear systems by taking the second-order perturbation into account [25].

ACKNOWLEDGMENTS

This work was supported by NSF of China under Grants No. 60578014 and No. 10775023, and the National Basic Research Program of China.
[14] The present analysis is for a level crossing; it can easily be generalized for an avoided level crossing.